Journal of Applied Mechanics and Technical Physics, Vol. 38, No. 6, 1997

SPONTANEOUS AXISYMMETRIC SWIRLING IN AN IDEALLY CONDUCTING FLUID IN A MAGNETIC FIELD

B. A. Lugovtsov

UDC 532.516+538.4

The problem of spontaneous swirling was considered in [1-7] and is as follows: can rotary motion occur in the absence of external rotation sources, i.e., under conditions where motion without rotation is realizable?

A more rigorous formulation of this problem was given in [5, 6]. The formulation proposed there ensures a strict control of the kinematic flow of the axial component of the angular momentum, which eliminates inflow of the rotating fluid in the flow region. The occurrence of rotary motion is regarded as a bifurcation of the initial axisymmetric flow as a result of the loss of stability against swirling flow [1].

In [5, 6], it is shown that the bifurcation "axisymmetric flow–rotationally symmetric flow" (and the corresponding plane analog of this transition [6]) does not occur for an arbitrary compressible fluid with a variable viscosity coefficient. In the case of the plane analog, this statement is also valid for a conducting fluid moving in the presence of a magnetic field. It has been shown [7] that for axisymmetric flow of a viscous incompressible fluid with a finite conductivity, axisymmetric spontaneous swirling is impossible if the meridian section of the flow region is simply connected.

For an ideally conducting fluid, the character of connectedness of the flow region is of no significance, because, in axisymmetric flows of such a fluid, the poloidal components of the magnetic field do not disappear in any case because of the frozen state.

In this paper, we give an example of the occurrence of spontaneous swirling in the initially axisymmetric flow of an inviscid, ideally conducting fluid in a magnetic field.

In the generally adopted notation (the fluid density $\rho = 1$) in the cylindrical coordinates r, φ, z, the equations that describe such flow are of the form

$$u_t + uu_r + wu_z - \frac{v^2}{r} + \left(p + \frac{1}{2}\mathbf{h}^2\right)_r = h_1 h_{1r} + h_3 h_{1r} - \frac{h^2}{r} + f_1; \tag{1}$$

$$w_t + uw_r + ww_z + \left(p + \frac{1}{2}\mathbf{h}^2\right)_z = h_1 h_{3r} + h_3 h_{3z} + f_3; \tag{2}$$

$$\Psi_t + u\Psi_r + w\Psi_z = 0, \quad h_1 = -\frac{1}{r}\Psi_z, \quad h_3 = \frac{1}{r}\Psi_r, \quad \mathbf{h} = \frac{\mathbf{H}}{\sqrt{4\pi}}. \tag{3}$$

These equations describe the poloidal (radial and axial) components of the velocity, $u = -(1/r)\psi_z$ and $w = (1/r)\psi_r$, and of the magnetic field, h_1 and h_3, where f_1 and f_3 are the corresponding components of the external mass forces.

The azimuthal components of the velocity, $v_\varphi = v$, and of the magnetic field, $h_\varphi = h$, fit the equations

$$v_t + uv_r + wv_z + \frac{uv}{r} = h_1 h_r + h_3 h_z + \frac{h_1 h}{r}; \tag{4}$$

$$h_t + uh_r + wh_z - \frac{uh}{r} = h_1 v_r + h_3 v_z - \frac{h_1 v}{r}. \tag{5}$$

On the boundary of the axisymmetric region D, the conditions $\mathbf{vn} = 0$ and $\mathbf{hn} = 0$ should be satisfied. Here $\mathbf{n} = (n_r, 0, n_z)$ is the external unit normal to the boundary of the flow region D.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 38, No. 6, pp. 29-31, November-December, 1997. Original article submitted December 30, 1996.

0021-8944/97/3806-0839 $18.00

The law of conservation of energy is satisfied for this system. We denote

$$\varepsilon_p = \frac{1}{2}(u^2 + w^2 + {h_1}^2 + {h_3}^2), \qquad \varepsilon_\varphi = \frac{1}{2}(v^2 + h^2).$$

Here ε_p is the energy density of the poloidal components of the velocity and the magnetic field, and ε_φ is the energy density of the azimuthal components of the velocity and of the magnetic field. By virtue of Eqs. (1)–(5) we have

$$\frac{d}{dt}\int_D (\varepsilon_p + \varepsilon_\varphi) r\, dr\, dz = \int_D (uf_1 + wf_3) r\, dr\, dz.$$

In this case, the following equalities are satisfied:

$$\frac{d}{dt}\int_D r\varepsilon_p\, dr\, dz = \int_D u(v^2 - h^2)\, dr\, dz + \int_D (uf_1 + wf_3) r\, dr\, dz,$$

$$\frac{d}{dt}\int_D r\varepsilon_\varphi\, dr\, dz = -\int_D u(v^2 - h^2)\, dr\, dz.$$

The last two relations show that, in axisymmetric flows, the poloidal and azimuthal components can exchange energy, in contrast to the plane analog of the flow considered, in which this exchange is absent and the corresponding energy components do not vary with time if the external forces are equal to zero.

However, the mere fact of exchange does not suggest the possibility of spontaneous swirling. This can be seen from the fact that, in the absence of a magnetic field, the exchange occurs, but spontaneous swirling is impossible, because, in this case, from Eq. (4) we obtain

$$\frac{d}{dt}\int_D r^3 v^2\, dr\, dz = 0.$$

As mentioned above, for spontaneous swirling to occur, a mechanism must exist that ensures counter gradient flow of the axial component of the angular momentum. We show that, in the presence of a magnetic field, this mechanism can lead to the initiation of spontaneous swirling.

We assume that, in a certain axisymmetric region D, there is a stationary solution of system (1)–(5) that satisfies the boundary conditions $\mathbf{vn} = 0$ and $\mathbf{hn} = 0$, so that $\psi_0(r,z) = \lambda\Psi_0(r,z)$, $v = 0$, and $h = 0$ for $f_1 = 0$ and $f_2 = 0$, where λ is a constant. Such flows exist. An example is the well-known Hill magnetohydrodynamic vortex with flow inside a sphere on which $\psi = \Psi = 0$.

We set $A = v + h$ and $B = v - h$. Then, for A and B from Eqs. (4) and (5) in a linear approximation we have

$$A_t + (\lambda - 1)h_{01}A_r + (\lambda - 1)h_{03}A_z + (\lambda + 1)\frac{h_{01}}{r}B = 0,$$

$$B_t + (\lambda + 1)h_{01}B_r + (\lambda + 1)h_{03}B_z + (\lambda - 1)\frac{h_{01}}{r}A = 0,$$

where h_{01} and h_{03} are the corresponding components of the steady magnetic field.

From these equations we have the following conservation law:

$$\frac{d}{dt}\int_D [(1 - \lambda)A^2 + (1 + \lambda)B^2] r\, dr\, dz = 0.$$

Hence it follows that, for $|\lambda| < 1$, i.e., for a sufficiently strong field, the initial flow (in a linear approximation) is stable against swirling.

Let $\lambda = 1$. If we now specify an infinitely small perturbation $v = v_0(\psi_0)$ and $h = 0$ for $t = 0$, the linear problem of the stability of the initial flow has the solution

$$A = v_0(\psi_0) + 2v_0(\psi_0)\frac{\psi_{0z}}{r^2}t, \qquad B = v_0(\psi_0)$$

and, hence,

$$v = v_0(\psi_0)(1 + \frac{\psi_{0z}}{r^2}t), \qquad h = v_0(\psi_0)\frac{\psi_{0z}}{r^2}t. \tag{6}$$

Thus, in this case, the initial steady flow is unstable and, as a result, a swirling flow occurs. If, under the same initial conditions, the initial poloidal components of the velocity and the magnetic field are maintained so that $\psi = \Psi = \psi_0$ by means of the external mass forces $f_1 = -(v^2 - h^2)/r$ and $f_3 = 0$, Eqs. (6) are an exact solution of system (1)–(5). Of course, the introduction of these forces is an artificial expedient, but, with no magnetic field, swirling is impossible with any external forces having only poloidal components.

This example shows that, in axisymmetric flows, the flow of the angular momentum related to the magnetic fields can lead to the occurrence of a mechanism that ensures counter gradient flow of the mechanical angular momentum, which maintains differential rotation, and, hence, spontaneous swirling can occur.

Note that the instability that is found is a new, previously unknown type of instability in magnetohydrodynamic flows.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 96-01-01771).

REFERENCES

1. M. A. Gol'dshtik, E. M. Zhdanova, and V. N. Shtern, "Spontaneous swirling of a buoyant jet," *Dokl. Akad. Nauk SSSR*, **277**, No. 4, 815–818 (1984).
2. M. A. Gol'dshtik, V. N. Shtern, and N. I. Yavorskii, *Viscous Flows with Paradoxical Properties* [in Russian], Nauka, Novosibirsk (1989).
3. M. A. Gol'dshtik and V. N. Shtern, "Turbulent vortex dynamo," *Prikl. Mat. Mekh.*, **53**, No. 4, 613–624 (1989).
4. A. M. Sagalakov and A. Yu. Yudintsev, "Three-dimensional self-oscillating magnetohydrodynamic flows of a liquid with a finite conductivity in an annular channel in the presence of a longitudinal magnetic field," *Magn. Hydrodyn.*, No. 1, 41–48 (1993).
5. B. A. Lugovtsov, "Is spontaneous swirling of axisymmetric flow possible?" *Prikl. Mekh. Tekh. Fiz.*, **35**, No. 2, 50–54 (1994).
6. Yu. G. Gubarev and B. A. Lugovtsov, "On spontaneous swirling in axisymmetric flows," *Prikl. Mekh. Tekh. Fiz.*, **36**, No. 4, 52–59 (1995).
7. B. A. Lugovtsov, "On spontaneous swirling in axisymmetric flows of a conducting fluid in a magnetic field," *Prikl. Mekh. Tekh. Fiz.*, **37**, No. 6, 35–43 (1996).